ON THE THEORY OF MOTION OF RIGID BODIES WITH FLUID-FILLED CAVITIES

(K TEORII DVIZHENIIA TVERDYKH TEL S POLOSTIAMI NAPOLNENNYMI ZHIDKOST'IU)

PMM, Vol. 30, No. 1, 1966, pp. 51-66

V.V. RUMIANTSEV
(Moscow)

(Received June 28, 1965)

The Lagrangian equations of motion are derived from the principle of least action in the Hamilton-Ostrogradskii form, for a rigid body with cavities, partially or completely filled with an ideal fluid possessing surface tension. First integrals of these equations are considered.

Further conditions derived from the equations of motion are these for which there exists an equilibrium or a stationary motion of the fluid-filled body, and which reduce to the extremal conditions (stationary state) of the potential energy Π or the altered potential energy of the system W. Previously [1] we gave the formulation of the stability problem of a rigid body with a fluid possessing surface tension, and the theorems which reduced the solution of the stability problem to the problem of finding the minimum of Π (or W). In the practically interesting cases, the problem of minimum W is solved by investigating the second variation $\delta^2 W$ the derivation of which is presented below.

The theorem on instability of equilibrium of fluid filled body is proved in the nonlinear formulation for the case, when the potential energy of the system does not have a minimum in the position of equilibrium.

1. Let us consider an absolutely rigid body with a cavity filled partially or fully with an ideal homogeneous incompressible fluid. The body and the fluid in it will be regarded as a single mechanical system and its motion with respect to a fixed (inertial) system of coordinates $O'x_1'x_2'x_3'$ will be investigated. In addition, we introduce a moving system of coordinates $Ox_1x_2x_3$, which is rigidly attached to the body with the origin at some point O in the body. The radius vector of an arbitrary point P_ν of the system relative to the point O' will be denoted by $\mathbf{r}_\nu'$, and relative to the point O by $\mathbf{r}_\nu$. The absolute velocity of the point P_ν can be represented in the form

$$\mathbf{v}_\nu = \mathbf{v}_0 + \boldsymbol{\omega} \times \mathbf{r}_\nu + \mathbf{u}_\nu$$

where $\mathbf{v}_0$ is the velocity vector of the point P, ω is the vector of the instantaneous angular velocity of the body, $\mathbf{u}_\nu = d\mathbf{r}_\nu / dt$ is the relative velocity vector of the point P_ν in its motion relative to the body. Obviously, for the points on a rigid $\mathbf{u}_\nu = 0$. The kinetic energy T of the system is composed from the kinetic energy T_1 of the body and T_2 of the fluid, where

$$T_1 = {}^1/_2 M_1 \mathbf{v}_0{}^2 + M_1 \mathbf{v}_0 \cdot (\boldsymbol{\omega} \times \mathbf{r}_1) + {}^1/_2 \boldsymbol{\omega} \cdot \Theta^{(1)} \cdot \boldsymbol{\omega}, \quad T_2 = \rho \int_\tau T^\circ \, d\tau \tag{1.1}$$

Let us define the notation: M_1 and $\mathbf{r}_1$ are the mass and the radius vector respectively of the center of mass of the body, $\Theta^{(1)}$ is the inertia tensor of the body at the point O, $T^0 = {}^1/_2 v^2$ is the density of the kinetic energy of the fluid, τ is the volume of the space $x_1 x_2 x_3$, occupied by the fluid at a given instant of time, ρ is the density of the fluid.

The volume τ is bounded by the walls σ_1 of the cavity which are in contact with the fluid at a given instant of time, and by the free surface S (for the case of partially filled cavity) the equation for which can be represented in the form

$$f(x_1, x_2, x_3, t) = 0 \tag{1.2}$$

The surface σ of the cavity walls consists, generally speaking, of the surface σ_1 which is in contact with the fluid at the given instant of time, and the surface σ_2 which is in contact with air. The boundary between these parts of the surface σ is the line l of intersection of the surfaces S, σ. If the free surface S of the fluid does not intersect with the walls σ of the cavity then, obviously, the line l does not exist. In the following, it will be assumed that the surface S is smooth or,that it consist of a finite number of smooth pieces of the surface.

The mass and motion of air in the cavity partially filled with a fluid will be neglected, regarding the air pressure p_0 as constant.

We shall further assume, that the considered rigid body with the fluid is constrained by certain ideal geometric bounds,or is free. The number of degrees of freedom for the body will be denoted by $n (n \leqslant 6)$.

The position of the system will be determined by the Lagrangian coordinates of the body q_j $(j = 1,\ldots,n)$ and the Cartesian coordinates of the fluid particles x_i $(i = 1,2,3)$. In this case the vectors $\mathbf{v}_0$ and $\boldsymbol{\omega}$ can be represented in the form of certain linear functions of the generalized velocities $q_j^\cdot$ with the coefficients depending on the generalized coordinates q_j. Utilizing these expressions and substituting them into the formulas (1.1), the kinetic energy of the body and the density of the fluid kinetic energy will be represented in the form of the functions of second order in $q_j^\cdot$ and u_i

$$T_1 = T_1(q_j^\cdot, q_j, t), \qquad T^\circ = T^\circ(q_j^\cdot, q_j, u_i, x_i, t)$$

The vector of the given active force applied to some point of the system will be denoted by $\mathbf{F}_\nu$. Among these forces will be the forces acting on the rigid body, mass forces acting on the fluid, and surface tension forces.

Following the concept of Gauss, we shall assume that the contact of two nonhomogeneous media r and s along a certain surface, will result in tensile forces whose potential will be equal to the product of the contact surface area and the coefficient of surface tension α_{rs}, dependent upon the nature of both media, and where, obviously, $\alpha_{rs} = \alpha_{sr}$. In the present case there are, generally speaking, three such nonhomogeneous media: the rigid body, the fluid, and the air, which will be ascribed the indices 1, 2 and 3 respectively. For simplicity we will denote $\alpha = \alpha_{23}$, $\alpha_1 = \alpha_{12}$, $\alpha_2 = \alpha_{13}$. In futures we shall assume these coefficients to be constant.

For the derivation of the equations of motion of a rigid body with a fluid we will utilize the principle of least action in the Hamilton-Ostrogradski form. Taking into account the condition of incompressibility of fluid, the principle can be written in the form [2]

$$\int_{t_0}^{t_1} \left(\delta T + \sum_\nu \mathbf{F}_\nu \cdot \delta \mathbf{r}_\nu' + \int_\tau p \operatorname{div} \delta \mathbf{r}_\nu' \, d\tau\right) dt = 0 \tag{1.3}$$

Here the symbol δ denotes the variation or a change of the corresponding quantity during a possible displacement (for $\delta t = 0$), where on the constant regions of integration

$$\delta \mathbf{r}_\nu' = 0 \qquad \text{for} \quad t = t_0, \; t = t_1 \tag{1.4}$$

$p\ (x_1,\ x_2,\ x_3,\ t)$ is the Lagrange multiplier which in the present case represents hydrodynamic pressure.

Taking the variation of the expression for the kinematic energy of the system

$$T = T_1 + \rho \int_\tau T^\circ \, d\tau$$

we obtain

$$\delta T = \sum_{j=1}^{n} \left(\frac{\partial T}{\partial q_j} \delta q_j + \frac{\partial T}{\partial \dot{q}_j} \delta \dot{q}_j \right) + \sum_{i=1}^{3} \rho \int_\tau \left(\frac{\partial T^\circ}{\partial x_i} \delta x_i + \frac{\partial T^\circ}{\partial u_i} \delta u_i\right) d\tau \tag{1.5}$$

Let us now sum all the terms of the virtual work applied to the active forces over the possible displacements of the system. Since for the points on the rigid body and fluid particles $\mathbf{r}' = \mathbf{r}'(q_j,\ x_i,\ t)$, then

$$\begin{gathered} \sum_\nu \mathbf{F}_\nu \cdot \delta \mathbf{r}_\nu' = \sum_{j=1}^{n} Q_j \delta q_j + \rho \int_\tau \sum_{i=1}^{3} F_i \delta x_i \, d\tau - \alpha \delta S - \\ - \alpha_1 \delta \sigma_1 - \alpha_2 \delta \sigma_2 - p_0 \int_S \mathbf{n} \cdot \delta_1 \mathbf{r} \, dS, \qquad Q_j = \sum_\nu \mathbf{F}_\nu \cdot \frac{\partial \mathbf{r}_\nu'}{\partial q_j} \end{gathered} \tag{1.6}$$

where Q_j ($j = 1, \ldots, n$) are the generalized forces, $\delta_1 \mathbf{r}$ is the variation of the radius vector $\mathbf{r} = x_1\mathbf{i}_1 + x_2\mathbf{i}_2 + x_3\mathbf{i}_3$ for fixed unit vectors of the moving axes $\mathbf{i}_s$, $\mathbf{n}$ is the unit vector along the external normal to the surface S.

The vector of possible fluid displacement relative to the rigid body $\delta_1\mathbf{r}$ will be regarded as a continuous differentiable function of the radius vector $\mathbf{r}$ satisfying the conditions of incompressibility in the region τ, impermeability of the walls σ, and of the conservation of fluid volume

$$\operatorname{div} \delta_1\mathbf{r} = 0, \qquad \mathbf{n}_\sigma \cdot \delta_1\mathbf{r} = 0, \qquad \int_S \mathbf{n}\cdot\delta_1\mathbf{r}\, dS = 0$$

Here $\mathbf{n}_\sigma$ is the normal to the surface σ. The variations of the free surface area S, the wetted area σ_1, and dry area σ_2 of the cavity surface σ for a possible displacement are

$$\delta S = \int_S 2H\delta\zeta\, dS + \int_l \delta\zeta_1\, dl, \qquad \delta\sigma_1 = -\delta\sigma_2 = \int_l \delta\zeta_2\, dl$$
$$\delta\zeta = \mathbf{n}\cdot\delta_1\mathbf{r}, \quad \delta\zeta_i = \mathbf{n}_i\cdot\delta_1\mathbf{r} \qquad (i = 1, 2) \qquad H = \frac{1}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \tag{1.7}$$

Here H is the average curvature of the surface, R_1 and R_2 are the main radii of curvature for the surface S at a given point, taken as positive if the center of curvature lies at the same side of the surface as the fluid, and negative otherwise; $\mathbf{n}_1$ and $\mathbf{n}_2$ are the unit vectors of the external normals to the contour l of the surfaces S and σ_1 located, respectively, at the tangent surfaces to these areas. The angle between the normals $\mathbf{n}_1$ and $\mathbf{n}_2$ will be denoted by θ. Assuming that in the neighborhood of the contour l the area of the cavity walls σ is sufficiently smooth and does not have sharp edges, we find that

$$\delta\zeta_1 = \delta\zeta_2 \cos\theta \tag{1.8}$$

Let us also consider the integral

$$\int_\tau p \operatorname{div} \delta\mathbf{r}'\, d\tau = \int_\tau p \operatorname{div} \delta_1\mathbf{r}\, d\tau$$

Since

$$p \operatorname{div} \delta_1\mathbf{r} = \operatorname{div}(p\delta_1\mathbf{r}) - \operatorname{grad} p\cdot\delta_1\mathbf{r}$$

Then using the Gauss-Ostrogradskii theorem we obtain

$$\int_\tau p \operatorname{div} \delta_1\mathbf{r}\, d\tau = \int_S p\mathbf{n}\cdot\delta_1\mathbf{r}\, dS - \int_\tau \operatorname{grad} p\cdot\delta_1\mathbf{r}\, d\tau \tag{1.9}$$

Substituting the expressions (1.5) to (1.9) into the principle (1.3) we have

$$\int_{t_0}^{t_1} \Big\{ \sum_{j=1}^{n} \left(\frac{\partial T}{\partial q_j^{\cdot}} \delta q_j^{\cdot} + \frac{\partial T}{\partial q_j} \delta q_j \right) + \sum_{i=1}^{3} \rho \int_{\tau} \left(\frac{\partial T^\circ}{\partial u_i} \delta u_i + \frac{\partial T^\circ}{\partial x_i} \delta x_i + F_i \delta x_i \right) d\tau +$$
$$+ \sum_{j=1}^{n} Q_j \delta q_j - \int_S (2\alpha H + p_0 - p)\, \delta\zeta\, dS - \int_l (\alpha \cos\theta + \alpha_1 - \alpha_2)\, \delta\zeta_2\, dl -$$
$$- \int_\tau \operatorname{grad} p \cdot \delta_1 \mathbf{r}\, d\tau \Big\}\, dt = 0$$

Integrating by parts the terms $(\partial T/\partial q_j^{\cdot})\delta q_j^{\cdot}$ and $(\partial T^\circ/\partial u_i)\delta u_i$ and taking into account the fact that the conditions (1.4) in the integration region are equivalent to the following:

$$\delta q_j = 0, \quad \delta x_i = 0 \qquad \text{for} \quad t = t_0,\ t = t_1$$

we obtain

$$\int_{t_0}^{t_1} \Big\{ \sum_{j=1}^{n} \left(- \frac{d}{dt} \frac{\partial T}{\partial q_j^{\cdot}} + \frac{\partial T}{\partial q_i} + Q_j \right) \delta q_j + \sum_{i=1}^{3} \rho \int_{\tau} \Big(- \frac{d}{dt} \frac{\partial T^\circ}{\partial u_i} + \frac{\partial T^\circ}{\partial x_i} + F_i -$$
$$- \frac{1}{\rho} \frac{\partial p}{\partial x_i} \Big) \delta x_i d\tau - \int_S (2\alpha H + p_0 - p)\, \delta\zeta\, dS - \int_l (\alpha \cos\theta + \alpha_1 - \alpha_2)\, \delta\zeta_2\, dl \Big\}\, dt = 0$$

Since δq_j and δx_i are independent, the equations of motion for the rigid body with a fluid are obtained in the Lagrangian form

$$\frac{d}{dt} \frac{\partial T}{\partial q_j^{\cdot}} - \frac{\partial T}{\partial q_j} = Q_j \qquad (j = 1, \ldots, n) \tag{1.10}$$
$$\frac{d}{dt} \frac{\partial T^\circ}{\partial u_i} - \frac{\partial T^\circ}{\partial x_i} = F_i - \frac{1}{\rho} \frac{\partial p}{\partial x_i} \qquad (i = 1, 2, 3) \tag{1.11}$$

as well as the boundary conditions for the pressure p on the free surface S

$$p = p_0 + 2\alpha H \tag{1.12}$$

and the boundary angle θ on the l contour

$$\cos\theta = (\alpha_2 - \alpha_1) / \alpha \tag{1.13}$$

To these equations and boundary conditions we add the equation for the incompressibility of the fluid

$$\operatorname{div} \mathbf{u} = 0 \tag{1.14}$$

as well as the kinematic conditions on the rigid walls σ_1

$$u_n = \mathbf{u} \cdot \mathbf{n} = 0 \tag{1.15}$$

and on the free surface S

$$\partial f/\partial t + \mathbf{u}\cdot\mathrm{grad}\, f = 0 \tag{1.16}$$

Thus, the study of motion for a rigid body with an ideal fluid in its cavities is reduced to the investigation of the simultaneous system of equations (1.10), (1.11) and (1.14) with the boundary conditions (1.12), (1.13), (1.15) and (1.16). Note that in the case when the forces of surface tension are neglected, the condition (1.12) becomes: $p = p_0$ on S, and the condition (1.13) vanishes. Condition (1.13) will thus be a consequence of less simple equations (see formula (3.4)) which should be integrated by taking into account the forces of surface tension in order to obtain the equation of the type (1.2). For the case when the fluid fills the cavity completely, the conditions (1.12), (1.13) and (1.16) are naturally excluded.

The equations (1.10) have the usual Lagrangian form. If it is assumed that $\rho = 0$, which corresponds to the case of no fluid in the cavity, then the equations (1.10) will represent the Lagrangian equations of motion for the rigid body alone.

Let us note that in the case when the active forces applied to the system have a force function $U(\mathbf{r}_\nu', t)$, i.e.

$$\mathbf{F}_\nu = \mathrm{grad}_{\mathbf{r}_\nu'} U$$

the generalized forces

$$Q_j = \sum_\nu \frac{\partial \mathbf{r}_\nu'}{\partial q_j}\cdot \mathrm{grad}_{\mathbf{r}_\nu'} U = \frac{\partial U}{\partial q_j} \qquad (j = 1, \ldots, n)$$

and the equations (1.10) assume the form

$$\frac{d}{dt}\frac{\partial L}{\partial \dot q_j} - \frac{\partial L}{\partial q_j} = 0 \qquad (j = 1, \ldots, n) \tag{1.17}$$

where $L = T + U$ is the Lagrangian functional. In the general case the force function of the active forces acting on the system is

$$U = U_1 + \rho \int_\tau U_2\, d\tau + U_2^* \qquad (U_2^* = -(\alpha S + \alpha_1\sigma_1 + \alpha_2\sigma_2))$$

Here $U_1(q_j, t)$ is the force function of the active forces applied to the rigid body, $U_2(q_j, x_i, t)$ is the force function of the mass forces acting on the fluid, U_2^* is the force function of the surface tension forces. The function U will in the following be considered continuous, and possessing continuous partial derivative along all coordinates. Denoting by $L^\circ = T^\circ + U_2$ the Lagrangian function for a unit mass of the fluid, the equations (1.11) can be expressed as

$$\frac{d}{dt}\frac{\partial L^\circ}{\partial u_i} - \frac{\partial L^\circ}{\partial x_i} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} \qquad (i = 1, 2, 3) \tag{1.18}$$

since in presence of potential forces

$$F_i = \frac{\partial U_2}{\partial x_i} \qquad (i = 1, 2, 3)$$

Equations (1.11) or (1.18) represent Euler's hydrodynamical equations. Indeed, it is easy to see that

$$\frac{\partial T^\circ}{\partial u_i} = v_i \quad (i = 1, 2, 3), \qquad \frac{\partial T^\circ}{\partial x_1} = \omega_3 v_2 - \omega_2 v_3 \qquad (123)$$

so that the equations (1.11) can be expressed in the form of a single vector equation

$$\frac{d\mathbf{v}}{dt} + \omega \times \mathbf{v} = \mathbf{F} - \frac{1}{\rho} \operatorname{grad} p \tag{1.19}$$

representing the Euler equations referred to the moving coordinate system.

Note that the presented derivation of the equations of motion is valid also for the case of the motion of the fluid in a fixed vessel. In this case if is only necessacy to put everywhere $q_j = q_j^{\cdot} = 0$ $(j = 1, \ldots, n)$.

The equations of motion of the ideal fluid will be of the form of equation (1.11) or (1.18) and (1.14) with the boundary conditions (1.12), (1.13), (1.15) and (1.16). At the same time the system of coordinates $Ox_1x_2x_3$, associated with the rigid body will be a fixed system, while the vector $\mathbf{u}$ will be the absolute fluid velocity, since $\mathbf{v}_0 = \omega = 0$.

It should be noted that the representation of the equations of motion in the form of Lagrange's equations presents the possibility of applying the well developed methods of analytical mechanics and control theory to the theory of motion of rigid bodies filled with a fluid.

Since the body with the fluid is regarded as a single mechanical system it was not necessary to determine the interaction forces between the rigid body and the fluid. In some cases, however, it is necessary to compute the forces exerted by the fluid and air in the cavity on the rigid body.

Let us find the expression for the generalized force of pressure of the fluid and air on the walls of the cavity

$$P_j = \int_\sigma p\mathbf{n} \cdot \frac{\partial \mathbf{r}'}{\partial q_j} d\sigma$$

Applying the Gauss-Ostrogradskii formula and taking into account that

$$\operatorname{div}\left(p \frac{\partial \mathbf{r}'}{\partial q_j}\right) = p \operatorname{div} \frac{\partial \mathbf{r}'}{\partial q_j} + \frac{\partial \mathbf{r}'}{\partial q_j} \cdot \operatorname{grad} p, \qquad \operatorname{div} \frac{\partial \mathbf{r}'}{\partial q_j} = 0$$

we obtain

$$P_j = \int_\tau \frac{\partial \mathbf{r}'}{\partial q_j} \cdot \operatorname{grad} p \, d\tau$$

since for air $p = p_0$. Replacing grad p by its equivalent from the Euler's equation, results in

$$P_j = \rho \int_\tau \frac{\partial \mathbf{r}'}{\partial q_j} \cdot \left(\mathbf{F} - \frac{d\mathbf{v}}{dt}\right) d\tau$$

where, by contrast with the equation (1.19), $d\mathbf{v} / dt$ is the absolute derivative of the vector $\mathbf{v}$ with respect to time t.

Let us consider the expression

$$\frac{\partial \mathbf{r}'}{\partial q_j} \cdot \frac{d\mathbf{v}}{dt} = \frac{d}{dt}\left(\mathbf{v} \cdot \frac{\partial \mathbf{r}'}{\partial q_j}\right) - \mathbf{v} \cdot \frac{d}{dt} \frac{\partial \mathbf{r}'}{\partial q_j}$$

It is easy to verify that

$$\frac{\partial \mathbf{v}}{\partial \dot{q}_j} = \frac{\partial \mathbf{r}'}{\partial q_j}, \qquad \frac{d}{dt} \frac{\partial \mathbf{r}'}{\partial q_j} = \frac{\partial \mathbf{v}}{\partial q_j}$$

and consequently

$$\frac{\partial \mathbf{r}'}{\partial q_j} \cdot \frac{d\mathbf{v}}{dt} = \frac{d}{dt}\left(\mathbf{v} \cdot \frac{\partial \mathbf{v}}{\partial \dot{q}_j}\right) - \mathbf{v} \cdot \frac{\partial \mathbf{v}}{\partial q_j} = \frac{d}{dt} \frac{\partial T^\circ}{\partial \dot{q}_j} - \frac{\partial T^\circ}{\partial q_j}$$

Substituting this expression into the formula for P_j we obtain

$$P_j = -\frac{d}{dt} \frac{\partial T_2}{\partial \dot{q}_j} + \frac{\partial T_2}{\partial q_j} + \rho \int_\tau \mathbf{F} \cdot \frac{\partial \mathbf{r}'}{\partial q_j} d\tau \tag{1.20}$$

Taking this into account, we can write the equation (1.10) in the form of equations of motion for a rigid body

$$\frac{d}{dt} \frac{\partial T_1}{\partial \dot{q}_j} - \frac{\partial T_1}{\partial q_j} = Q_j^{(1)} + P_j \qquad (j = 1, \ldots, n) \tag{1.21}$$

which is under the influence of the given applied forces and of the pressure forces exerted by the fluid and air on the cavity walls. Here

$$Q_j^{(1)} = \sum_\nu{}_{(1)} \mathbf{F}_\nu \cdot \frac{\partial \mathbf{r}_\nu'}{\partial q_j}$$

where the summation occurs only along the points on the rigid body.

Equations (1.21) should be investigated along with the equations (1.11) and (1.14) with the corresponding boundary conditions.

2. The equations of motion of a rigid body with a fluid permit first integrals under specific conditions [2].

In the following it will be assumed that the forces applied to the system and the motion of the rigid body are continuous, while the motion of the fluid occurs in its entirety so that the fluid particle coordinates remain continuous functions of their initial values and time.

It is known that if the constraints imposed on the mechanical system are not dependent on time explicitly, and the given active forces possess a stationary force function, then there exists an energy integral.

Indeed, let the given condition be satisfied. Let us investigate the Lagrangian functional $L(q_j, q_j^{\cdot}, u_i, x_i) = T + U$. Its derivative, on the strength of (1.17) and (1.18), is

$$\frac{dL}{dt} = \frac{d}{dt}\left[\sum_{j=1}^{n} \frac{\partial L}{\partial q_j^{\cdot}} q_j^{\cdot} + \rho \int_{\tau} \sum_{i=1}^{3} \frac{\partial L^{\circ}}{\partial u_i} u_i d\tau\right] + \sum_{i=1}^{3} \int_{\tau} \frac{\partial p}{\partial x_i} u_i d\tau + \frac{dU_2^*}{dt} \qquad (2.1)$$

Utilizing the equation for incompressibility (1.14) and applying the Gauss-Ostrogradskii theorem, we find, by taking into account the boundary conditions (1.12) and (1.15)

$$\sum_{i=1}^{3} \int_{\tau} \frac{\partial p}{\partial x_i} u_i d\tau = \int_{S} (p_0 + 2H\alpha)\, u_n dS$$

But in view of the formulas

$$\frac{dS}{dt} = \int_{S} 2Hu_n dS + \int_{l} u_1 dl, \qquad \frac{d\sigma_1}{dt} = -\frac{d\sigma_1}{dt} = \int_{l} u_2 dl, \qquad u_1 = u_2 \cos\theta \qquad (2.2)$$

which can be obtained from the formulas (1.7) and (1.8) by replacing the symbol δ by d/dt, where $u_i = \mathbf{u}\cdot\mathbf{n}_i$ $(i = 1, 2)$, we obtain, with (1.13) taken into account,

$$\frac{dU_2^*}{dt} = -\alpha \int_{S} 2Hu_n dS \qquad (2.3)$$

Since, by the conservation of the volume of the fluid

$$\int_{S} u_n dS = 0$$

then obviously

$$\sum_{i=1}^{3} \int_{\tau} \frac{\partial p}{\partial x_i} u_i d\tau + \frac{dU_2^*}{dt} = 0 \qquad (2.4)$$

Equation (2.1) then yields the energy integral

$$-L + \sum_{j=1}^{n} \frac{\partial L}{\partial q_j^{\cdot}} q_j^{\cdot} + \rho \int_{\tau} \sum_{i=1}^{3} \frac{\partial L^{\circ}}{\partial u_i} u_i d\tau = \text{const} \qquad (2.5)$$

It is easy to verify by direct computation that

$$\sum_{j=1}^{n} \frac{\partial L}{\partial q_j^{\cdot}} q_j^{\cdot} + \sum_{i=1}^{3} \rho \int_{\tau} \frac{\partial L^{\circ}}{\partial u_i} u_i d\tau = 2T \qquad (2.6)$$

so that the integral (2.5) has the usual form of the conservation of energy integral

$$T + \Pi = \text{const} \qquad (\Pi = \Pi_1 + \rho\int_\tau \Pi_2 d\tau + \Pi_2^* = -U + \text{const})$$

Here Π denotes the potential energy of the system.

Let further, the coordinates q_α ($\alpha = k+1, \ldots, n$) be cyclic, i.e.

$$\partial L/\partial q_\alpha = 0 \qquad (\alpha = k+1, \ldots, n) \tag{2.7}$$

In this case the equations (1.17) yield the cyclic first integrals

$$\partial L/\partial q_\alpha^{\cdot} = \text{const} \qquad (\alpha = k+1, \ldots, n) \tag{2.8}$$

The momentum and second momentum integrals [2] are related to the integrals of the type (2.8).

Applying the Routh's method of ignorable cyclical coordinates it is possible to lower the order of the system of differential equations (1.17) for the motion of the rigid body with a fluid. Let us consider, for example, the case when the functional L is independent of the rotation angle of the body about some fixed straight line $O'x_3'$. Let us introduce the system of coordinates $O'\xi_1\xi_2x_3'$, capable of rotating about the axis x_3' with the angular velocity ω of the rigid body in its motion about this axis. The angle of rotation q_n of the body about x_3' can be assumed to be the angle between the axes x_1' and ξ_1. The kinetic energy of the system is represented in the form [3]

$$T = T^{(1)} + q_n^{\cdot} G_{x_3'}^{(1)} + {}^1\!/_2 q_n^{\cdot 2} J \tag{2.9}$$

where $T^{(1)}$ and $G_{x_3'}^{(1)}$ are the kinetic energy and the projection on the x_3' axis of the second momentum of the system in its motion relative to the system of coordinates $O'\xi_1\xi_2x_3'$; J denotes the moment of inertia about the x_3' axis. The first integral of the form (2.8)

$$\frac{\partial L}{\partial q_n^{\cdot}} = G_{x_3'}^{(1)} + q_n^{\cdot} J = k = \text{const}$$

corresponds to the cyclical coordinate q_n.

Hence, we find

$$q_n^{\cdot} = (k - G_{x_3'}^{(1)}) / J$$

and exclude this quantity from the Routh functional $R = L - q_n^{\cdot} k$.

Consequently we obtain by the usual method [4] the equation of motion for a rigid body with a fluid in form of the Routh equation

$$\frac{d}{dt}\frac{\partial R}{\partial q_j^{\cdot}} - \frac{\partial R}{\partial q_j} = 0 \qquad (j = 1, \ldots, n-1) \tag{2.10}$$

If the function of R does not depend on time explicitely, then these equations along with the equations (1.18) permit a first energy integral

$$R_2 - R_0 = \text{const} \tag{2.11}$$

where R_s is the homogeneous and of the degree s with respect to the velocities $q_j^\cdot$, part of the functional $R = R_2 + R_1 + R_0$. It is easy to verify that

$$R_2 = T^{(2)} = T^{(1)} - G_{x_3'}^{(1)2} / 2J, \qquad R_0 = -\Pi - k^2 / 2J \tag{2.12}$$

so that the integral (2.11) will become

$$T^{(2)} + \Pi + k^2 / 2J = \text{const}$$

3. The equations of motion for a rigid body with a fluid permit, under certain conditions, solutions describing equilibrium or a stationary motion of the system. Let us determine these conditions assuming that the given active forces applied to the system are the potential ones. Assuming that $q_j^\cdot = 0$, $u_i = 0$ $(i = 1,2,3,\ j = 1,\ldots,n)$ in (1.17) we obtain

$$\partial L / \partial q_j = \partial U / \partial q_j = 0 \qquad (j = 1, \ldots, n) \tag{3.1}$$

Consequently, at the equilibrium position of the rigid body with a fluid the force function U, or the potential energy of the system Π, have extremal (stationary) values, i.e.

$$\delta U = \delta \Pi = 0 \tag{3.2}$$

Equations (3.1) define the coordinates of the rigid body at equilibrium. Without loss of generality it can be assumed that $q_j = 0$ $(j = 1, \ldots, n)$ will be the roots of the equation (3.1). The fluid at equilibrium occupies the region τ_0, bounded by the cavity walls and the free surface. The equation for the latter $U_2 - p/\rho = \text{const}$ is found by integrating the equation (1.18) for $u_i = 0$, $q_j = q_j^\cdot = 0$. Taking into account the boundary condition (1.12) it becomes

$$\rho U_2 - 2\alpha H = \text{const} \tag{3.3}$$

The constant in the right hand part of the equation (3.3) can be determined from the known values of the average curvature H and the force function U_2 for some point of the free surface. From the differential geometry [5] it is known that

$$2H = \frac{EN - 2FM + GL}{EG - F^2} \tag{3.4}$$

where E, F, G, and L, M, N are the coefficients of the first and second differential Gauss forms for the surface. Thus, equation (3.3) is a nonlinear partial differential equation.

The form of the free surface S of the fluid at equilibrium is determined by integrating the equation (3.3) with the boundary condition (1.13) taken into account. The integration of this equation is, generally speaking, quite difficult [6]. For the case when the surface tension forces can be neglected, the equation (3.3) becomes the boundary equation for the free surface of the fluid at equilibrium.

Let us assume now that among the coordinates q_j of the rigid body there is a cyclic coordinate q_n. Then the equations of motion for a rigid body with a fluid permit a particular solution in which all non-cyclic coordinates q_j $(j=1,\ldots,n-1)$ remain constant as well as the velocity $q_n^{\cdot}$, corresponding to the cyclic coordinate q_n, while all non-cyclical; velocities of the body $q_j^{\cdot}$ and the relative velocities u_i of the fluid particles are zero. Such a solution describes a steady motion representing a uniform rotation of the entire system as a single rigid body with an angular velocity $q_n^{\cdot}$ about the axis x_3'. From the equation (2.10) for a fixed value of the constant $k=k_0$ follow the equations

$$\partial R_0 / \partial q_j = 0 \quad (j=1,\ldots,n-1) \tag{3.5}$$

for the coordinates q_j of the rigid body in steady motion. Consequently, for the steady motion the altered potential energy of the system $W=-R_0=k_0^2/2J+\Pi$ attains the extremal (stationary) value

$$\delta W = 0 \tag{3.6}$$

Let us assume that $q_j=0$ $(j=1,\ldots,n-1)$ satisfy equations (3.5). The fluid in this case occupies the region τ_0, bounded by the walls of the cavity and the free surface S_0. The equation for the latter will be found by integrating (1.18) with

$$q_j = q_j^{\cdot} = 0 \quad (j=1,\ldots,n-1), \quad q_n^{\cdot} = \omega, \quad u_i = 0 \quad (i=1,2,3)$$

and by taking into account the boundary conditions (1.12)

$$\rho\,[U_2 + \omega^2 (x_1'^2 + x_2'^2)/2] - 2\alpha H = \mathrm{const} \tag{3.7}$$

Equation (3.7) differs from the equation (3.3) only in the form of the force function; everything previously said about equation (3.3) is also valid for (3.7).

Paper [1] gives the definition of the stability of motion for a rigid body with a fluid possessing surface tension and proves the theorems reducing the question of stability of a steady motion (or equilibrium) to the minimum problem of the altered potential energy W (or the potential energy Π) of the system. For the cases of practical interest the minimum problem of W (or Π) can be solved by investigating the second variation $\delta^2 W$ (or $\delta^2\Pi$). In taking the variation the fluid volume τ must be conserved, and let

$$W = \frac{k_0^2}{2J} + \Pi + \lambda \int_\tau d\tau \quad (\lambda = \mathrm{const})$$

The fisrt variation of W is found easily as

$$\delta W = \sum_{j=1}^{n-1} \frac{\partial W}{\partial q_j}\delta q_j - \int_S \Big[\rho U_2 + \frac{1}{2}\frac{k_0^2}{J^2}\rho\,(x_1'^2 + x_2'^2) - 2\alpha H - \lambda\Big]\delta\zeta\, dS + \int_l (\alpha\cos\theta + \alpha_1 - \alpha_2)\,\delta\zeta_2\, dl \tag{3.8}$$

It is assumed that the quantity $\delta\zeta$ is a continuous differentiable function of the curvilinear coordinates u, and v of a point on the free surface of the fluid possessing continuous partial derivatives with respect to u, and v and satisfying the condition of conservation of the fluid volume

$$\int_{S_0} \delta\zeta \, dS = 0 \tag{3.9}$$

In view of the independence of δq_j and $\delta\zeta$ we obtain the equations (3.5) from the equation $\delta W = 0$ for the coordinates of the rigid body in a stationary motion and the equation (3.7) of the free surfaces S_0 of the fluid in the same motion as well as the condition (1.13) for the boundary angle.

For simplicity it will be further assumed that the area of 'equilibrium' S_0 is simply connected and smooth. The line of intersection of the surface S_0 with the walls of the cavity σ, if this line exists, will be denoted by l_0 and will be regarded as piece-wise smooth. The unit vector of the tangent $\mathbf{e}$ to the contour l_0 is oriented so that in going around l_0 the region S_0 remains on the left, for the observer located along the outward normal n to S_0. The moving system of coordinates $Ox_1x_2x_3$, fixed to the rigid body will be selected so that in the unperturbed position of equilibrium the Ox_3 axis would coincide with the $O'x_3'$ axis. For brevity we introduce the notation

$$\Phi(q_j, x_i) \equiv \rho\, [U_2(x_1', x_2', x_3') + {}^1/_2\, \omega^2 (x_1'^2 + x_2'^2)]_{x_i' \to q_j,\, x_i} \tag{3.10}$$

after which the equation (3.7) can be expressed as

$$\Phi(0, x_i) - 2\alpha H = \lambda = \text{const} \tag{3.11}$$

Taking the variation of equation (3.8) we obtain the following expression for the second variation of W in the neighborhood of the considered stationary motion of the system

$$\delta^2 W = \sum_{i,\, j=1}^{n-1} \left(\frac{\partial^2 W}{\partial q_i \partial q_j}\right)_0 \delta q_i \delta q_j - 2\int_{S_0} \sum_{j=1}^{n-1} \left(\frac{\partial \Phi}{\partial q_j}\right)_0 \delta q_j \delta\zeta \, dS +$$
$$+ \frac{\omega^2}{J_0}\, [2\Delta_1 J \Delta_2 J + (\Delta_2 J)^2] - \int_{S_0} \delta\, (\Phi - 2H\alpha - \lambda)\, \delta\zeta dS + \int_{l_0} \alpha\delta \cos\theta\delta\zeta_2 \, dl \tag{3.12}$$

Here

$$\Delta_1 J = \sum_{j=1}^{n-1} \left(\frac{\partial J}{\partial q_j}\right)_0 \delta q_j, \qquad \Delta_2 J = \rho \int_{S_0} (x_1'^2 + x_2'^2)_{x_i' \to q_j,\, x_i} \delta\zeta \, dS \tag{3.13}$$

where the index O denotes the value of the corresponding quantity for the unperturbed motion.

Let us obtain the explicit expressions for the terms under the integral signs in the last two components of the right hand side of the inequality (3.12). We consider the first one.

Between the perturbed and unperturbed surfaces of the fluid it is possible, apparently, to establish a correspondence along the normals to the unperturbed surface. Also, the variation of the average curvature is determined from the formula [5]

$$\delta H = -(2H^2 - K)\,\delta\zeta - {}^1/_2\,\Delta^\circ\delta\zeta \tag{3.14}$$

Where

$$K = 1 / R_1R_2, \qquad \Delta^\circ\delta\zeta = (E\delta\zeta_{22} - 2F\delta\zeta_{12} + G\delta\zeta_{11}) / (EG - F^2)$$

denote respectively the Gauss curvature of the surface S_0 and the second differential parameter of Beltrami, $\delta\zeta_{ij}$ are the covariant second derivatives of the function $\delta\zeta$. Utilizing (3.14) and (3.9) we obtain

$$\int\limits_{S_0} \delta(\Phi - 2H\alpha - \lambda)\,\delta\zeta\,dS = \int\limits_{S_0}\left[\left(\frac{\partial\Phi}{\partial n}\right)_0 \delta\zeta + 2\alpha(2H^2 - K)\,\delta\zeta + \alpha\Delta^\circ\delta\zeta\right]\delta\zeta dS \tag{3.15}$$

According to Green's formula [5]

$$\int\limits_{S_0} \delta\zeta\Delta^\circ\delta\zeta dS = \int\limits_{l_0} \delta\zeta\,\frac{d\delta\zeta}{ds_1}\,dl - \int\limits_{S_0} \nabla^\circ\delta\zeta dS \tag{3.16}$$

where

$$\nabla^\circ\delta\zeta = [E\,(\delta\zeta)_v{}^2 - 2F\,(\delta\zeta)_u(\delta\zeta)_v + G\,(\delta\zeta)_u{}^2]/(EG - F^2)$$

denotes the first differential Beltrami parameter, $(\delta\zeta)_u$, and $(\delta\zeta)_v$ are the derivatives of the functions $\delta\zeta$ with respect to u, and v respectively, $d\delta\zeta / ds_1$ is the derivative of $\delta\zeta$ along the outward normal $\mathbf{n}_1$ to the contour l_0 of the surface S_0. On the strength of (3.16) the equality (3.15) can be expressed as

$$\int\limits_{S_0} \delta(\Phi - 2H\alpha - \lambda)\,\delta\zeta dS = \int\limits_{S_0}\left\{\left[\left(\frac{\partial\Phi}{\partial n}\right)_v + 2\alpha(2H^2 - K)\right](\delta\zeta)^2 - \alpha\nabla^\circ\delta\zeta\right\}dS + \alpha\int\limits_{l_0}\delta\zeta\,\frac{d\delta\zeta}{ds_1}\,dl \tag{3.17}$$

We find now

$$\delta\cos\theta = \mathbf{n}_2\cdot\delta\mathbf{n}_1 + \mathbf{n}_1\cdot\delta\mathbf{n}_2.$$

On the surface S_0 in the neighborhood of the contour l_0, the coordinate lines u, and v will be represented by l_0 and the curves orthogonal to it. The unit vectors of the tangents to these curves are $\mathbf{e}$ and $\mathbf{n}_1$. Also

$$\mathbf{n}_1 = \mathbf{r}_v / \sqrt{G}, \qquad \delta\mathbf{n}_1 = \delta\mathbf{r}_v / \sqrt{G} - \mathbf{r}_v\delta\sqrt{G}/G$$

Assuming that for the points on the contour l_0

$$\delta\mathbf{r} = \delta\xi\mathbf{r}_v + \delta\zeta\mathbf{n}$$

we find

$$\delta \mathbf{r}_v = (-G_u\delta\xi - 2M\delta\zeta)\,\mathbf{r}_u / 2E + [(\delta\xi)_v + (G_v\delta\xi - 2N\delta\zeta)/2G]\,/\,\mathbf{r}_v + [(\delta\zeta)_v + N\delta\xi]\,\mathbf{n}$$

$$\delta \sqrt{G} = \mathbf{r}_v \cdot \delta\mathbf{r}_v / \sqrt{G} = [(\delta\xi)_v + (G_v\delta\xi - 2N\delta\zeta)/2G]\sqrt{G}$$

so that

$$\delta\mathbf{n}_1 = (-G_u\delta\xi - 2M\delta\zeta)\,\mathbf{r}_u / 2E\sqrt{G} + [(\delta\zeta)_v + N\delta\xi]\,\mathbf{n}/\sqrt{G}$$

Taking into account that the vector $\mathbf{n}_2$ is orthogonal to the vector $\mathbf{r}_u = \sqrt{E}\mathbf{e}$, we find

$$\mathbf{n}_2 \cdot \delta\mathbf{n}_1 = [(\delta\zeta)_v / \sqrt{G} + N\sqrt{G}\delta\xi / G]\,\mathbf{n}\cdot\mathbf{n}_2$$

According to Meusnier's formula $N / G = 1/R_{n_1}$, where R_{n_1} is the radius of curvature of the cross-section to the normal surface S_0 in the direction $\mathbf{n}_{,1}$ an element of the arc of which is $ds_1 = \sqrt{G}dv$.

Since $\sqrt{G}\delta\xi = \delta\zeta_1$, $\mathbf{n}\cdot\mathbf{n}_2 = \sin\theta$,

$$\mathbf{n}_2\cdot\delta\mathbf{n}_1 = (d\delta\zeta / ds_1 + \delta\zeta_1 / R_{n_1})\sin\theta \tag{3.18}$$

Similarly we find

$$\mathbf{n}_1\cdot\delta\mathbf{n}_2 = -\sin\theta\delta\zeta_2 / R_{n_2} \tag{3.19}$$

where R_{n_2} denotes the radius of curvature of the cross-section normal to the walls of the cavity σ in the direction $\mathbf{n}_2$. Summing the equalities (3.18) and (3.19) we obtain

$$\delta\cos\theta = (d\delta\zeta / ds_1 + \delta\zeta_1 / R_{n_1} - \delta\zeta_2 / R_{n_2})\sin\theta \tag{3.20}$$

Note that the formula (3.20) allows one to obtain the boundary condition for the functions $\delta\zeta$ on the contour l_0 of the surface S_0 for the case of the linear treatment of the problem. Indeed, from the condition (1.13) for the boundary angle formed by the free surface of the fluid and the cavity walls, it follows that in the first approximation we have on the contour l_0,

$$d\delta\zeta / ds_1 = \delta\zeta_2 / R_{n_2} - \delta\zeta_1 / R_{n_1} \quad \text{for} \quad \sin\theta \neq 0 \tag{3.21}$$

When $\sin\theta = 0$, it is clear that on l_0

$$\delta\zeta = \sin\theta\,\delta\zeta_2 = 0 \tag{3.22}$$

On the basis of the equalities (3.12), (3.17) and (3.20) as well as (1.8), the expression for the second variation of the altered potential energy of the system becomes

$$\delta^2 W = \sum_{i,\,j=1}^{n-1}\left(\frac{\partial^2 W}{\partial q_i \partial q_j}\right)_0 \delta q_i \delta q_j + \frac{\omega^2}{J_0}[2\Delta_1 J\Delta_\circ J + (\Delta_2 J)^2] -$$

$$-\int_{S_0}\left\{2\sum_{j=1}^{n-1}\left(\frac{\partial\Phi}{\partial q_j}\right)_0 \delta q_j\delta\zeta + \left[\left(\frac{\partial\Phi}{\partial n}\right)_0 + 2\alpha(2H^2 - K)\right](\delta\zeta)^2 - \alpha\nabla^\circ\delta\zeta\right\} dS + \tag{3.23}$$

$$+ \alpha \int_{l_0} \left(\frac{\delta\zeta_1}{R_{n_1}} - \frac{\delta\zeta_2}{R_{n_2}}\right) \delta\zeta_2 \sin\theta \, dl \tag{3.23}$$

while the function Φ (q_j, x_i) and the quantity $\Delta_1 J$ and $\Delta_2 J$ are defined by the equalities (3.10) and (3.13).

Note that it is possible to obtain, from the expression (3.23), the particular formula for the second variation of the potential energy system (for $\omega = 0$):

$$\delta^2\Pi = \sum_{i,j=1}^{n} \left(\frac{\partial^2\Pi}{\partial q_i \partial q_j}\right)_0 \delta q_i \delta q_j - \int_{S_0} \left\{2\rho \sum_{j=1}^{n} \left(\frac{\partial U_2}{\partial q_j}\right)_0 \delta q_j \delta\zeta + \left[\rho \left(\frac{\partial U_2}{\partial n}\right)_0 + \right.\right.$$
$$\left.\left. + 2\alpha(2H^2 - K)\right](\delta\zeta)^2 - \alpha\nabla^\circ\delta\zeta\right\} dS + \alpha \int_{l_0} (\delta\zeta_1 / R_{n_1} - \delta\zeta_2 / R_{n_2}) \delta\zeta_2 \sin\theta \, dl \tag{3.24}$$

From the principles of the variational calculus it is known that if the altered potential energy of the system W is to have a minimum it is necessary that its second variation $\delta^2 W$ be non-negative, and it is sufficient that the second variation be strongly positive for all sufficiently small absolute values of q_j and δx_i, which satisfy the condition that the condition of incompressibility and (3.9) are not simultaneously equal to zero.

The problem of determining the conditions for strong positiveness of the second varition $\delta^2 W$ can,be a known method [7],be reduced to the problem of finding the criterion of positiveness of the smallest eigen-value of the corresponding boundary value problem.

4. Let us investigate the question of the character of equilibrium of a rigid body with a fluid in the case when the potential energy of the system does not have a minimum at the position of equilibrium. For the system with a finite number of degrees of freedom the proofs of the Lagrange reduction theorem are given by Liapunov and Chetaev [4].

An analogous theorem can be proved for a rigid body with a fluid. Let us restrict ourselves to the case when in the neighborhood of the position of equilibrium the potential energy of the system is of the form

$$\Pi = \Pi^{(2)} + \Pi^{(3)} + \dots$$

where $\Pi^{(k)}$ is a k-th order homogenous functional for the deviation of the system from the equilibrium position.

Theorem 4.1. If for an arbitrarily small neighborhood of the position of equilibrium for a rigid body with a fluid the potential energy Π of the system can assume negative values and, if at the same time, the signs of the expressions $\Pi^{(2)} + \Pi^{(3)} + \dots$ and $2\Pi^{(2)} + 3\Pi^{(3)} + \dots$ are defined by the quadratic functional $\Pi^{(2)}$, then the position of equilibrium is unstable.

Proof. Assume that in the position of equilibrium the coordinates of the rigid body are $q_j = 0$ $(j = 1, \dots, n)$, the coordinates of fluid particles are $x_i = x_{i0}$ $(i = 1, 2, 3)$ and that the fluid fills the volume τ_0. Let at the position of equilibrium the system potential energy be $\Pi = 0$ and be not a minimum; at the neighborhood of equilibrium there exists a region where $\Pi < 0$. Disturbing the equilibrium position of the system momentarily, we investigate the perturbed motion which is described by the equations (1.17), (1.18) and (1.14) with the corresponding boundary conditions.

The radius vector of the displacement from equilibrium of the fluid particle relative to the rigid body will be denoted by $\Delta \mathbf{r} = \mathbf{r} - \mathbf{r}_0$. Obviously,

$$\Delta \mathbf{r} = \Delta_0 \mathbf{r} + \int_{t_0}^{t} \mathbf{u}\, dt \tag{4.1}$$

where $\Delta_0 \mathbf{r}$ is the radius vector of the initial fluid displacement, and where in view of incompressibility, div $\Delta_0 \mathbf{r} = 0$. Then, in view of (1.14) it follows from (4.1) that

$$\operatorname{div} \Delta \mathbf{r} = 0 \tag{4.2}$$

The slope of the perturbed fluid surface relative to the unperturbed surface is characterized by the partial derivatives n_u and n_v of the function $n(u, v) = \Delta\zeta$. The maximum of $|n_u|$, and $|n_v|$ will be denoted by ∇ which we shall call the inclination of the perturbed surface relative to the unperturbed one.

Let us consider the functional [4]

$$V = -(T + \Pi)\left(\sum_{j=1}^{n} \frac{\partial L}{\partial q_j^{\cdot}} q_j + \sum_{i=1}^{3} \rho \int_{\tau} \frac{\partial T^{\circ}}{\partial u_i} \Delta x_i \, d\tau\right) \tag{4.3}$$

In a sufficiently small neighborhood of the equilibrium position, i.e. in the region of small absolute values of the corrdinates q_j and Δx_i, the inclinations ∇, and the velocities $q_j^{\cdot}$ and u_i we choose an infinite-dimensional region (C) existing for arbitrarily small absolute quantities q_j, Δx_i, ∇, $q_j^{\cdot}$ and u_i and defined by the simultaneous inequalities

$$T + \Pi < 0, \qquad \sum_{j=1}^{n} \frac{\partial L}{\partial q_j^{\cdot}} q_j + \sum_{i=1}^{3} \rho \int_{\tau} \frac{\partial T^{\circ}}{\partial u_i} \Delta x_i d\tau > 0 \tag{4.4}$$

The existence of the first of these inequalities for sufficiently small $q_j^{\cdot}$, and u_i is obvious in view of the conditions of the theorem for the nonexistence of the minimum of Π. At the points in the neighborhood of the position of equilibrium where this inequality is fulfilled, the velocities $q_j^{\cdot}$, and u_i can always be selected of such a sign that the second inequality in (4.4) is fulfilled.

In the region (C) the functional V is, obviously, bounded, i.e. there exists such a positive number N that in the region (C)

$$|V| < N \tag{4.5}$$

The time derivative of the functional V is, in view of the equations of perturbed motion, equal to

$$V^{\cdot} = -(T + \Pi)\left\{\sum_{j=1}^{n} \left(\frac{\partial L}{\partial q_j} q_j + \frac{\partial L}{\partial q_j^{\cdot}} q_j^{\cdot}\right) + \right.$$

$$+\sum_{i=1}^{3}\rho\int_{\tau}\left[\left(\frac{\partial T^\circ}{\partial x_i}+\frac{\partial U_2}{\partial x_i}-\frac{1}{\rho}\frac{\partial p}{\partial x_i}\right)\Delta x_i+\frac{\partial T^\circ}{\partial u_i}u_i\right]d\tau\Big\}$$

Taking into account the relation

$$\sum_{j=1}^{n}\frac{\partial L}{\partial q_j^{\cdot}}q_j^{\cdot}+\sum_{i=1}^{3}\rho\int_{\tau}\frac{\partial T^\circ}{\partial u_i}u_i d\tau=2T \tag{4.6}$$

the validity of which can be verified by direct computation, the expression for $V^{\cdot}$ can be written as

$$V^{\cdot}=-(T+\Pi)\Big\{2T+\sum_{j=1}^{n}\frac{\partial T}{\partial q_j}q_j+\sum_{i=1}^{3}\rho\int_{\tau}\left(\frac{\partial T^\circ}{\partial x_i}+\frac{\partial U_2}{\partial x_i}-\frac{1}{\rho}\frac{\partial p}{\partial x_i}\right)\Delta x_i d\tau+\sum_{j=1}^{n}\frac{\partial U}{\partial q_i}q_j\Big\} \tag{4.7}$$

As the consequence of the fact that the constraints imposed on the rigid body are assumed not to depend explicitely on time, the kinetic energy T is positive-definite with respect to $q_j^{\cdot}$ and u_i. For sufficiently small absolute values of the quantities q_j and Δx_i, the expression

$$2T+\sum_{j=1}^{n}\frac{\partial T}{\partial q_j}q_j+\sum_{i=1}^{3}\rho\int_{\tau}\frac{\partial T^\circ}{\partial x_i}\Delta x_i\,d\tau$$

will, because of continuity, also be positive-definite with respect to $q_j^{\cdot}$ and u_i.

Let us now consider the remaining terms in the expression (4.7). Utilizing the formula (3.24) and taking into account that $\Pi=\Pi^{(2)}+\Pi^{(3)}+\ldots$, where $\Pi^{(2)}=\delta^2\Pi/2$, $\Pi^{(3)}=\delta^3\Pi/3!$, we find

$$\frac{\partial U}{\partial q_j}=-\frac{\partial \Pi}{\partial q_j}=-\sum_{i=1}^{n}\left(\frac{\partial^2\Pi}{\partial q_j\partial q_i}\right)_0 q_i+\rho\int_{S_0}\left(\frac{\partial U_2}{\partial q_j}\right)_0 ndS$$

so that

$$\sum_{j=1}^{n}\frac{\partial U}{\partial q_j}q_j=-\sum_{i,j=1}^{n}\left(\frac{\partial^2\Pi}{\partial q_i\partial q_j}\right)_0 q_iq_j+\rho\int_{S_0}\sum_{j=1}^{n}\left(\frac{\partial U_2}{\partial q_j}\right)_0 q_j ndS+\ldots \tag{4.8}$$

Further, in view of (4.2) and (1.12) we obtain

$$\sum_{i=1}^{3}\rho\int_{\tau}\frac{\partial U_2}{\partial x_i}\Delta x_i\,d\tau=\rho\int_{S_1}\left[U_2(0,x_i)+\sum_{j=1}^{n}\left(\frac{\partial U_2}{\partial q_j}\right)_0 q_j+\left(\frac{\partial U_2}{\partial n}\right)_0 n\right]ndS+\ldots$$

$$\sum_{i=1}^{3}\int_{\tau}\frac{\partial p}{\partial x_i}\Delta x_i\,d\tau=\alpha\int_{S_1}[2H-2(2H^2-K)n-\Delta^\circ n]\,ndS$$

Applying Green's formula (3.16) results in

$$\sum_{i=1}^{3}\int_{\tau}\frac{\partial p}{\partial x_i}\Delta x_i\,d\tau=\alpha\int_{S_0}[2Hn-2(2H^2-K)n^2+\nabla^\circ n]\,dS-\alpha\int_{l_0}n\frac{dn}{ds_1}\,dl$$

Thus, with the conditions (3.21) and (3.11) taken into account, we obtain

$$\sum_{j=1}^{n}\frac{\partial U}{\partial q_j}q_j+\sum_{i=1}^{3}\int_{\tau}\left(\rho\frac{\partial U_2}{\partial x_i}-\frac{\partial p}{\partial x_i}\right)\Delta x_i\,d\tau=-\sum_{i,\,j=1}^{n}\left(\frac{\partial^2\Pi}{\partial q_i\partial q_j}\right)_0 q_iq_j+$$

$$+\int_{S_0}\left[2\rho\sum_{j=1}^{n}\left(\frac{\partial U_2}{\partial q_j}\right)_0 q_jn+\rho\left(\frac{\partial U_2}{\partial n}\right)_0 n^2+2\alpha(2H^2-K)n^2-\alpha\nabla^\circ n\right]dS+$$

$$+\alpha\int_{l_0}\left(\frac{1}{R_{n_2}}-\frac{\cos\theta}{R_{n_1}}\right)n_2^2\sin\theta\,dl+\ldots=-2\Pi^{(2)}+\ldots\qquad(n_2=\Delta\zeta_2)$$

where the dots indicate terms of the type- $3\Pi^{(3)}+\ldots$ According to (4.7) we find

$$V^{\cdot}=-(T+\Pi)\left\{2T+\sum_{j=1}^{n}\frac{\partial T}{\partial q_j}q_j+\rho\int_{\tau}\sum_{i=1}^{3}\frac{\partial T^\circ}{\partial x_i}\Delta x_i\,d\tau-2\Pi^{(2)}+\ldots\right\}\qquad(4.9)$$

Since in the region (C) the potential energy is negative, while the signs of Π and of the expression $2\Pi^{(2)}+3\Pi^{(3)}+\ldots$ are determined by the terms of second order $\Pi^{(2)}<0$, then in the region (C) this expression is negative definite: $2\Pi^{(2)}+3\Pi^{(3)}+\ldots<0$.

Consequently, in the region (C) the derivative of the functional V is positive definite with respect to q_j, Δx_j, $q_j^{\cdot}$, and u_i, ∇.

Also the positive definiteness of the functional $V^{\cdot}$ in the region of positiveness of the functional V is determined analogously to the sign-definiteness of the function [4] in the region $V>0$. Selecting the initial perturbations q_{j0}, $\Delta_0 x_i$, ∇_0, $q_{j0}^{\cdot}$ and u_{i0} in the region (C) arbitrarily small so that the initial value $V_0>0$, we find from the equation

$$V=V_0+\int_{t_0}^{t}V^{\cdot}\,dt\qquad(4.10)$$

that in the course of time the inequality (4.5) will be violated, which proves the instability of the position of equilibrium.

Theorem 4.2. If in the position of equilibrium of a rigid body with a fluid potential energy of the system $\Pi=\Pi^{(2)}+\Pi^{(3)}+\ldots$ does not have a minimum and this is established from its second variation of $\Pi^{(2)}$ without the necessity of considering higher order terms, then the equilibrium position is unstable.

The proof can be presented by an almost identical repetition of the proof given by Chetaev [8] for the analogous case of a system with a finite number of degrees of freedom.

In an infinite-dimensional space of the variables q_j, Δx_i, n_u and n_v we consider the hypersphere with the center at the position of equilibrium and an arbitrarily small radius $\varepsilon>0$. On this hypersphere the continuous functional Π assumes for some values of a_s of the considered variables, its smallest value Π_0. In accordance with the assumption about the potential energy Π this smallest value will be negative and is determined by the functional $\Pi^{(2)}$. Consequently, Π_0 will be of second order of smallness in comparison with ε. Let the region which is being studied for instability of the perturbed motions, be defined by the inequalities

$$|q_j|<l, \quad |\Delta x_i|<l, \quad |q_j^{\cdot}|<l, \quad |u_i|<l, \quad |\nabla|<l \tag{4.11}$$

for an arbitrarily small positive l. It will be assumed that the quantities l and ε are related by some definite number, which may be very large.

Under such an assumption about the choice of l and ε, the expression

$$2\Pi_0 + \Pi^{(3)} + 2\Pi^{(4)} + \ldots \tag{4.12}$$

in the region (4.11) will undoubtedly be negative. In the perturbed motion the initial values of our variables will be a_s, and the initial values of all velocities will be assumed zero. Such a perturbed motion will occur in accordance with the law of kinetic energy $T + \Pi = \Pi_0$ and, consequently, will occur in the region defined by the inequality

$$\Pi_0 - \Pi \geqslant 0 \tag{4.13}$$

Let us consider the functional

$$V = \sum_{j=1}^{n} \frac{\partial L}{\partial q_j^{\cdot}} q_j + \rho \int_{\tau} \sum_{i=1}^{3} \frac{\partial T^{\circ}}{\partial u_i} \Delta x_i \, d\tau \tag{4.14}$$

and its time derivative which in view of the equations of perturbed motion (1.17), (1.18) and (1.14) is

$$V^{\cdot} = 2T + \sum_{j=1}^{n} \frac{\partial T}{\partial q_j} q_j + \rho \int_{\tau} \sum_{i=1}^{3} \frac{\partial T^{\circ}}{\partial x_i} \Delta x_i \, d\tau - 2\Pi^{(2)} + \ldots \tag{4.15}$$

For the considered perturbed motion occurring in the region (4.13), the derivative $V^{\cdot}$ will be positive. Indeed, as was shown above, the expression

$$2T + \sum_{j=1}^{n} \frac{\partial T}{\partial q_j} q_j + \rho \int_{\tau} \sum_{i=1}^{3} \frac{\partial T^{\circ}}{\partial x_i} \Delta x_i \, d\tau$$

for sufficiently small l will be positive-definite with respect to $q_j^{\cdot}$ and u_i, and

$$-(2\Pi^{(2)} + 3\Pi^{(3)} + \ldots) > 0 \tag{4.16}$$

in view of (4.12) and (4.13). Let us denote by l' the minimum of the expression in the left hand side of the inequality (4.16) inside the region defined by the inequalities (4.11) and (4.13). From the equation (4.10), where in the given case $V_0 = 0$, we conclude that $V > l'(t - t_0)$, as long as the variables do not violate the inequalities (4.11) during the interval of t_0 to t. Consequently, with the course of time the inequality (4.5) will be violated. The theorem is proved.

It is worth noting that, in the linear formulation, the reduction of the Lagrange theorem for a rigid body with a fluid was proved by Krein [9], in whose work the surface tension was not considered. In the nonlinear formulation the reduction of the Lagrange theorem for solid bodies was considered by A.A. Movchan [10] with the assumption that the potentials of external and internal forces are homogeneous of order *m*.

BIBLIOGRAPHY

1. Rumiantsev, V.V., Ob ustoichivosti dvizheniia tverdogo tela s zhidkost'iu, obladaiushche poverkhnostnym natiazheniem (On the Stability of Motion of a Rigid Body with a Fluid having a Surface Tension), *PMM*, Vol. 28, No. 4 1964.

2. Rumiantsev, V.V., Uravneniia dvizheniia tverdogo tela, imeiushchego polosti, ne polnost'iu napolnennye zhidkost'iu (The Equations of Motion for a Rigid Body having Cavities Incompletely Filled with a Fluid), *PMM*, Vol. 18, No. 6 1954.

3. Rumiantsev, V.V., Ob ustoichivosti ustanovivshikhsia dvizhenii tverdykh tel s polostiami, napolnennymi zhidkost'iu (On the Stability of Steady Motions of Rigid Bodies with Cavities Filled with Fluids), *PMM*, Vol. 26, No. 6 1962.

4. Chetaev, N.G., Ustoichivost' dvizheniia (Stability of Motion), Izd. 'Nauka', 1965, Third edition.

5. Bliashke, V., Differentsial'naia geometriia i geometricheskie osnovy teorii otnositel' nosti Einshteina (Differential Geometry and Geometrical Foundations of Einstein's Theory of Relativity), ONTI, 1935.

6. Beliaeva, M.A., Myshkis, A.D., and Tiuptsov A.D., Gidrostatika v slabykh gravitatsionnykh poliakh. I. Ravnovesnye formy poverkhnosti zhidkosti (Hydrostatics in Weak Gravitational Fields. I. Equilibrium Forms of a Fluid Surface), Izv. AN USSR, Mekhanika i mashinostroeie, No. 5 1964.

7. Courant, R., and Hilbert, D., Metody matematicheskoi fiziki (Methods of Mathematical Physics), M.-L., Gostekhizdat, 1945.

8. Chetaev, N.G., O neustoichivosti ravnovesiia, kogda silovaia funktsiia ne est' maksimum (On the Instability of Equilibrium when the Force Function is not Maximum), Uch. zap. Kazansk. un-ta, Vol. 98, kn. 9, Matematika, No. 3 1938.

9. Krein, S.G. and Moiseev, N.N., O kolebaniakh tverdogo tela, soderzhashchego zhidkost' so svobodnoi poverkhnost'iu (On the Oscillations of a Rigid Body Containing a Fluid with a Free Surface), *PMM*, 1957, Vol. 21, No. 2.

10. Movchan, A.A., Ob ustoichivosti protsessov deformirovaniia sploshnykh tel (On the Stability of Deformation Processes of Rigid Bodies), Arch. Mech. Stosowanej, Vol. 5, No. 15 1963.

Translated by V.A.C.